Optimal Control of Quaternion Propagation Errors in Spacecraft Navigation

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I. Introduction

IN this Note, it will be shown that optimal control techniques[1] can be employed to drive the numerical error (truncation, roundoff, commutation) in computing the quaternion vector to zero. It is also interesting to note that the normalization of the quaternion can be carried out by suitable choice of a performance index, which can be optimized. The method of normalization presented here can also be easily implemented if one simulates the quaternion differential equation on an analog computer.

Section II deals with the derivation of the error model. Section III presents the derivation of the control algorithm. The simulation results for validating the control algorithm are given in Sec. IV. The modified dynamic model for attitude determination is given in Sec. V.

II. Error Equations for Quaternion Propagation

The error equations used here are derived from from the theoretical development by Friedland.[2] Accordingly, the attitude motion of a strapdown system with reference to a nonrotating coordinate system is given by

$$\dot{q}_t(t) = \tfrac{1}{2}\Omega[w_t(t)]q_t(t); \quad q_t(t_0) = q_{t_0} \tag{1}$$

with $q_t^T q_t = 1$. $q_t(t)$ is a four-component vector. Assuming that $q_c(t)$ is the computed quaternion, the errors δq can be defined as

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$$\delta q = q_t - q_c \tag{2}$$

Similarly, if w_c is the computed angular velocity and w_t is the true angular velocity, then δw can be defined as

$$\delta w = w_t - w_c \tag{3}$$

q_c is governed by

$$\dot{q}_c(t) = \tfrac{1}{2}\Omega[w_c(t)]q_c(t) + u(t); \; q_c(t_0) = q_{c_0} \tag{4}$$

with $q_c^T q_c \leq 1$. $u(t)$ represents the control signals to be designed for driving δq to zero. $\Omega(w)$ is given by

$$\Omega(w) = \begin{bmatrix} 0 & w_3 & -w_2 & w_1 \\ -w_3 & 0 & w_1 & w_2 \\ w_2 & -w_1 & 0 & w_3 \\ -w_1 & -w_2 & -w_3 & 0 \end{bmatrix} \tag{5}$$

where w_1, w_2, and w_3 are the projections of the instantaneous angular velocity of the reference axes in the body.

Differentiating Eq. (2) and making use of Eq. (3), one gets

$$\delta\dot{q} = \tfrac{1}{2}\Omega(w_t(t))q_t(t) - \tfrac{1}{2}\Omega(w_t - \delta w)q_c(t) - u(t) \tag{6}$$

Equation (6) has been simplified in Ref. 2 and is given for convenience:

$$\delta\dot{q} = \tfrac{1}{2}\Omega(w_t)\delta q + \tfrac{1}{2}G(q_t)\delta w - u(t)_1, \quad \delta q(t_0) = \delta q_0 \tag{7}$$

Equation (7) denotes the error state equation. Assuming that the error state is avilable in each computational step, then a state variable feedback (u) can be given to drive the error δq to zero. $G(q)$ is given by

$$G(q) = \begin{bmatrix} q_4 & -q_3 & q \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \tag{8}$$

III. Optimal Control

For our purposes, without any loss of generality, δw can be assumed as zero. Then Eq. (7) can be rewritten as

$$\delta\dot{q} = \tfrac{1}{2}\Omega(w)\delta q - u, \quad \delta q(t_0) = \delta q_0 \tag{9}$$

The control signals $u(t)$ can be designed by minimizing a quadratic performance index of the type given below:

$$J(u) = \tfrac{1}{2}\int_{t_0}^{t_f} [\delta q^T Q \delta q + u^T R u]\,dt + \tfrac{1}{2}\delta q^T S \delta q \,|_{t=t_f} \tag{10}$$

where Q and S are assumed as at least positive semidefinitive constant weightage matrices and R as a positive definite constant matrix, and (t_0, t_f) defines the control interval.

The Hamiltonian H $(\delta q, u, \lambda)$ can be defined as

$$H(\delta q, u, \lambda) = \tfrac{1}{2}[\delta q^T Q \delta q + u^T R u) + \lambda^T[\tfrac{1}{2}\Omega(w)\delta q - u) \tag{11}$$

The necessary conditions of optimal control[1] are given by

$$\left.\frac{\partial H}{\partial u}\right|_{u=u^*} Ru^* - \lambda = 0, \quad u^* = R^{-1}\lambda \tag{12}$$

$$\frac{\partial H}{\partial(\delta q)} = -\dot{\lambda} = Q\delta q + \tfrac{1}{2}\Omega^T(w)\lambda \tag{13}$$

The transversality condition is given by

$$\lambda(t_f) = \frac{\partial}{\partial(\delta q)}[\tfrac{1}{2}\delta q^T S \delta q] = S\delta q \tag{14}$$

Equations (9), (13), and (14) constitute the solution of a two-point boundary value problem. The augmented state and adjoint equations can be rewritten as

$$\begin{bmatrix} \delta\dot{q} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}\Omega(w) & R^{-1} \\ -Q & -\tfrac{1}{2}\Omega^T(w) \end{bmatrix} \begin{bmatrix} \delta q \\ \lambda \end{bmatrix} \tag{15}$$

Since Eq. (15) is linear, it is possible to transform the two-point boundary-value problem to a terminal boundary value problem. Assume that

$$\lambda = P\delta q \tag{16}$$

where P is a symmetric positive defintie matrix. From Eq. (12), it follows that

$$u^* = R^{-1}P\delta q \tag{17}$$

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Differentiating Eq. (16), one gets

$$\dot{\lambda} = \dot{P}\delta q + P\delta\dot{q} \tag{18}$$

Equation (18) can be simplified using Eqs. (9) and (13) as follows:

$$-Q\delta q - ½\Omega^T(w)P\delta q = \dot{P}\delta q + P(½\Omega(w)\delta q - R^{-1}P\delta q) \tag{19}$$

For a general $\delta q \neq 0$,

$$\dot{P} = -½\Omega^T(w)P - ½P\Omega(w) + PR^{-1}P - Q \tag{20}$$

which is the matrix Riccati equation. From Eqs. (16) and (14), it follows, at the terminal time,

$$P(t_f) = S \tag{21}$$

Equation (20) can be solved backward from $t=t_f$ until $t=t_0$. The optimal control u^* can be obtained as

$$u^* = R^{-1}P\delta q = K\delta q \tag{22}$$

with $K=R^{-1}P$. The computed quaternions can be found from Eq. (4) as

$$\dot{q}_c(t) = ½\Omega[w_c(t)]q_c(t) + K(q_t - q_c) \tag{23}$$

Equation (23) can be used to normalize the quaternion. It is interesting to recall that Mitchell and Rogers[3] have employed derivative constraints in their simulation for satisfying the norm constraint. Only a single gain factor k has been used by them instead of a matrix of optimized control gains by us as given in Eq. (23). Results of Ref. 3 have also been further verified with practical test data in Ref. 4.

Adopting Eq. (23) is beset with practical difficulties because it needs the true quaternion vector q_t, which is normally not available. If, at each step, the following approximation is made, then this difficulty can be overcome.[5] Assuming

$$q_t \cong \frac{q_c}{\|q_c\|} \tag{24}$$

where $\|q_c\| = (q_{1c}^2 + q_{2c}^2 + q_{3c}^2 + q_{4c}^2)^{½}$, Eq. (23) can be rewritten as

$$\dot{q}_c(t) = ½\Omega[w_c(t)]q_c(t) + K\left(\frac{q_c}{\|q_c\|} - q_c\right)$$

which can be further simplified as

$$\dot{q}_c(t) = ½\Omega[w_c(t)]q_c(t) + K\left[\frac{1-\|q_c\|}{\|q_c\|}\right]q_c \tag{25}$$

Then, Eq. (25) can be easily implemented.

IV. Simulation Results

The algorithm has been simulated using the VAX 11/780 digital computer by assuming sinusoidal body rates experienced in all three axes of a satellite. Assuming an Earth-pointing satellite with an orbital period of 90 min, and angular vibration of magnitude equal to 0.5 arc min and frequency of 10 rad/s in all axes, the body rates can be generated as follows:

$$w_1 = \frac{1}{120}\frac{\pi}{180}\sin(10t)$$

$$w_2 = \frac{1}{15}\frac{\pi}{180} + \frac{\pi}{180}\cdot\frac{1}{120}\sin\left(10t + \frac{2\pi}{3}\right)$$

$$w_3 = \frac{1}{120}\frac{\pi}{180}\sin\left(10t + \frac{4\pi}{3}\right) \tag{26}$$

Two sets of weightage matrices have been assumed for the optimization with $Q=I$, $R=I$, and $Q=2I$, $R=I$. All the elements of S have been assumed as unity in both cases.

The matrix Riccati equation has been solved backward from $t_f = 10$ s, and it has been noticed that the P matrix attains steady state. The steady-state gains are summarized in Table 1. The time variation of K as a function of time is shown in Fig. 1 for case II.

Equation (25) has been simulated assuming different $\delta_q(t_0)$, both positive and negative and using the steady-state control gains summarized in Table 1. In all the cases simulated, the quantity $E = [1 - q_c^T q_c]$ has been found to converge to a very small quantity of the order of 10^{-12} from an initially large value of the order of $|E| = 0{\cdot}5$. Figure 2 presents the variation of E as a function of time for different cases simulated. Figure 3 gives the computed quaternion vector q_c.

V. Dynamic Model for Attitude Determination

In previous studies, the normalization has been carried out after estimating the quaternion vector. It can be of interest to

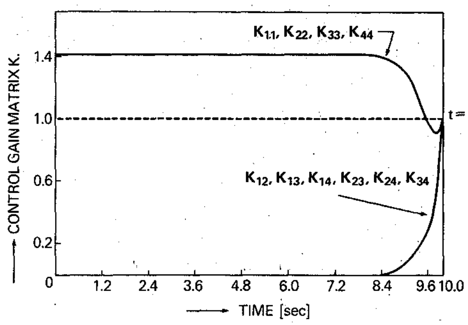

Fig. 1 Control gain matrix.

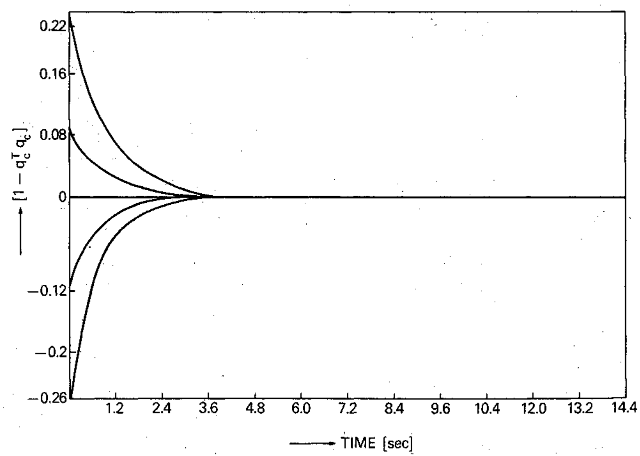

Fig. 2 Normalization error.

Table 1 Steady state gains

Case	$K11$	$K12$	$K13$	$K14$	$K22$	$K23$	$K24$	$K33$	$K34$	$K44$
I	1	0.16×10^{-8}	0.16×10^{-8}	0.16×10^{-8}	1	0.16×10^{-8}	0.16×10^{-8}	1	0.16×10^{-8}	1
II	1.4142×10^{-12}	0.54×10^{-12}	0.54×10^{-12}	0.54×10^{-12}	1.4142	0.54×10^{-12}	0.54×10^{-12}	1.4142	0.54×10^{-12}	1.4142

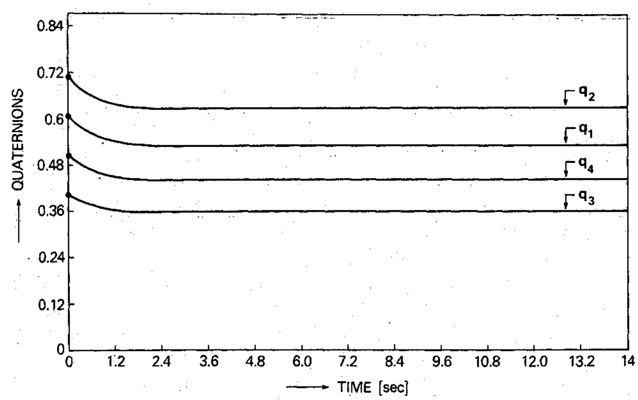

Fig. 3 Quaternions.

study the performance of the estimator in the presence of the feedback control that maintains the normalization. The following model, which follows Eq. (25), can be used for such purposes:

$$\dot{q} = \left[\tfrac{1}{2}\Omega(\omega) + K\left(\frac{1-\|q\|}{\|q\|}\right)\right] q \tag{27}$$

$$\omega = g - b - \eta \tag{28}$$

where g is the gyro output and b is the gyro drift rate to be estimated along with q. η denotes the gyro noise. Equation (27) can be used for propagation, and the star tracker data can be utilized to identify the gyro drift and update q as it is proposed in Ref. 6.

VI. Summary

Quaternion normalization has been successfully carried out using optimal control techniques. A matrix Riccati equation results for the computation of the gain matrix. Simulation results indicate that a high precision of the order of 10^{-12} can be achieved by this technique in meeting the constraint $q^T q = 1$.

The effect of normalization on the update equations needs further study. The modified model given in Eq. (27) can be utilized for this purpose. The scheme can be implemented without increasing very much the computer loading because K an be precomputed and stored.

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Book Announcements

CHANKONG, V., Khon Kaen University, and **HAIMES, Y.Y.**, Case Western Reserve University, ***Multiobjective Decision Making:* Theory and Methodology**, North-Holland, New York, 1983, 406 pages. $45.00.

Purpose: This book is intended for graduate students, researchers, and engineers interested in solving multiobjective optimization problems. Undergraduate-level understanding of set theory, matrices, probability, linear programming, and mathematic analysis is required.

Contents: Elements of multiobjective decision problems. Fundamentals: selected background topics. Utility theory. Vector optimization theory. Assessment methodologies. Methods for generating noninferior solutions. Noninteractive and interactive multiobjective progamming methods. The surrogate worth trade-off method and its extensions. Comparative evaluation and comments. Appendix. Author index. Subject index.

DOWNTON, A.C., University of Southampton, ***Computers and Microprocessors Components and Systems,*** Van Nostrand Reinhold (UK) Co. Ltd., 1984, 182 pages. $11.50.

Purpose: This book is intended as a beginning text for a lower-level undergraduate course and, as such, may be of interest to engineers who want to learn something about the details of computer operation. The goal of the text is to present an integrated approach to computers and microprocessors which places equal emphasis on components and systems, application and design.

Contents: Computers and microprocessors. Memory structure and architecture. Data representation in computers. Data processing. Input and output interfaces. Instruction size and addressing modes. Programming computers and microprocessors. Computer systems. System software. Microprocessor application design example. Appendixes. References. Answers to exercises and problems. Index.